ARITHMETIC PROPERTIES OF HARMONIC MODULAR FORMS OF WEIGHT ONE (Automorphic forms and automorphic L-functions)

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1. Introduction

A distinguished role within the theory of holomorphic modular forms is played by those having weight one. The Deligne-Serre theorem [2] identifies the L-function $L(s,f)$ of a newform f of weight one with the Artin L-function an irreducible odd two-dimensional representation of the Galois group G of a normal extension $K/\mathbb{Q}$. It is natural to seek arithmetic interpretations of the Fourier coefficients of a harmonic modular form of weight one. Here I will report on some recent joint work with Yingkun Li on this problem; see [3] for a detailed treatment and proofs.

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2. Harmonic modular forms

A harmonic modular form is a Maass form of weight $k\in\frac{1}{2}\mathbb{Z}$ for $\Gamma_0(M)$ that is annihilated by the weight k Laplacian and that is allowed polar-type singularities in the cusps. To such a form F is associated the weight $2-k$ weakly holomorphic form

$$\xi_kF(z)=2iy^k\overline{\partial_{\bar{z}}F(z)}.$$

The weight k Laplacian Δ_k can be written as

$$\Delta_k=\xi_{2-k}\circ\xi_k.$$

A special class of harmonic forms have Fourier expansions (necessariy unique) of the form

$$F(z)=\sum_{n\geq n_0}c^+(n)q^n-\sum_{n\geq 0}c(n)\beta_k(n,y)q^{-n}. \tag{1}$$

Here $q=e^{2\pi iz}$ with $z=x+iy\in\mathcal{H}$, the upper half-plane, and $\beta_k(n,y)$ is given for $n<0$ by

$$\beta_k(n,y)=\int_y^\infty e^{-4\pi nt}t^{-k}dt$$

while for $k\neq 1$ we have $\beta_k(0,y)=y^{1-k}/(k-1)$ and $\beta_1(0,y)=-\log y$. For such F the Fourier expansion of $\xi_k(F)$ is simply

$$\xi_k(F)=\sum_{n\geq 0}c(n)q^n.$$

We shall refer to the $c^+(n)$ as the *holomorphic Fourier coefficients* of $F(z)$.

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Prototypical examples are provided by Eisenstein series. When $k = 2$ we have for the full modular group the harmonic form

$$E_2^*(z) = 8\pi \sum_{n\geq 0} \sigma(n)q^n + y^{-1},$$

for which $\xi_2(E_2^*) = 1$. Here $\sigma(n) = \sum_{m|n} m$ for $n > 0$ and we set $\sigma(0) = -1/24$. When $k = 3/2$ we have Zagier's Eisenstein series for $\Gamma_0(4)$:

$$E_{3/2}^*(z) = 16\pi \sum_{n\geq 0} H(n)q^n - \sum_{n\in\mathbb{Z}} \beta_{3/2}(n^2, y)q^{-n^2},$$

for which $\xi_{3/2}(E_{3/2}^*) = \theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}$, the weight 1/2 Jacobi theta series. Here $H(n)$ is the Hurwitz class number with $H(0) = -1/12$. It is interesting to observe that in these (non-standard!) normalizations, the holomorphic Fourier coefficients of $E_2^*(z)$ and $E_{3/2}^*(z)$ are transcendental. In general the holomorphic Fourier coefficients $c^+(n)$ from (1) are not well understood and have been the focus of quite a lot of recent research, especially when $k = 1/2$ (see the Introduction of [3] for references).

The existence of a harmonic form F of weight $k < 2$ whose image $\xi_k(F)$ is a given cusp form of weight $2 - k$ is known in many cases. One method is to construct them using Poincaré series or their analytic continuations. This is the method we use in [3]. Another approach is given in [1]. In any case, such F is not unique since we can add to it any weakly holomorphic form f and $\xi_k(F + f) = \xi_k(F)$. In particular, the holomorphic Fourier coefficients of F are not determined by $\xi_k(F)$.

3. Weight one

We now turn to the case of interest to us, namely the self-dual case of weight $k = 1$. The Riemann-Roch theorem does not apply when $k = 2 - k$, and the existence of cusp forms is a subtle issue. We will provide some evidence that the holomorphic Fourier coefficients of a harmonic form F of weight one with $\xi_1(F)$ a newform, which is also of weight one and is associated to K by the Deligne-Serre theorem, contain arithmetic information about K. So far we have only been able to provide proofs in the dihedral case, but numerical evidence for more general results is promising.

An interesting harmonic modular form of weight one was constructed by Kudla, Rapoport and Yang [7]. Suppose that $M = p > 3$ is a prime with $p \equiv 3 \pmod 4$ and that $\chi_p(\cdot) = (\frac{\cdot}{p})$ is the Legendre symbol. Let

$$E_1(z) = \tfrac{1}{2}H(p) + \sum_{n\geq 1} R_p(n)q^n \tag{2}$$

be Hecke's Eisenstein series of weight one for $\Gamma_0(p)$ with character χ_p, where for $n > 0$

$$R_p(n) = \sum_{m|n} \chi_p(m). \tag{3}$$

It follows from [7] that for $n > 0$

$$R_p^+(n) = -(\log p)\mathrm{ord}_p(n)R_p(n) - \sum_{\chi_p(q)=-1} \log q(\mathrm{ord}_q(n) + 1)R_p(n/q) \tag{4}$$

gives the n-th holomorphic Fourier coefficient of a harmonic form of weight one whose image under ξ_1 is $E_1(z)$. This harmonic form is constructed using the s-derivative of the non-holomorphic Hecke-Eisenstein series of weight one. Its holomorphic Fourier coefficients $R_p^+(n)$ vanish when $n < 0$ and $R^+(0)$ can be given explicitly.

Our interest here is in the holomorphic Fourier coefficients of harmonic forms associated to newforms. We will continue to assume that $M = p > 3$ is a prime. To each $\mathcal{A} \in \mathrm{Cl}(F)$, the class group of F, one can associate a theta series $\vartheta_{\mathcal{A}}(z)$ defined by

$$\vartheta_{\mathcal{A}}(z) := \tfrac{1}{2} + \sum_{\substack{\mathfrak{a}\subset \mathcal{O}_F\\ [\mathfrak{a}]\in\mathcal{A}}} q^{\mathrm{N}(\mathfrak{a})} = \sum_{n\geq 0} r_{\mathcal{A}}(n) q^n.$$

Hecke showed that $\vartheta_{\mathcal{A}}(z) \in M_1(p,\chi_p)$, the space of weight one holomorphic modular forms for $\Gamma_0(p)$ with character χ_p. Let ψ be a character of $\mathrm{Cl}(F)$ and consider $g_\psi(z) \in M_1(p,\chi_p)$ defined by

$$g_\psi(z) := \sum_{\mathcal{A}\in\mathrm{Cl}(F)} \psi(\mathcal{A})\vartheta_{\mathcal{A}}(z) = \sum_{n\geq 0} r_\psi(n) q^n.$$

When $\psi = \psi_0$ is the trivial character, the form $g_{\psi_0}(z)$ is just $E_1(z)$ from (2), as a consequence of Dirichlet's fundamental formula

$$R_p(n) = \sum_{\mathcal{A}\in\mathrm{Cl}(F)} r_{\mathcal{A}}(n). \tag{5}$$

Otherwise, $g_\psi(z)$ is a newform in $S_1(p,\chi_p)$, the subspace of $M_1(p,\chi_p)$ consisting of cusp forms.

The following result shows that the holomorphic Fourier coefficients of certain harmonic modular forms of weight one whose image under ξ_1 is $g_\psi(z)$ can be expressed in terms of logarithms of algebraic numbers in H. It is to be noted that these harmonic forms have poles in the cusps and are not in general unique. Let H be the Hilbert class field of F with ring of integers $\mathcal{O}_H$ and denote by $\sigma_{\mathcal{C}} \in \mathrm{Gal}(H/F)$ the element associated to the class $\mathcal{C} \in \mathrm{Cl}(F)$ via Artin's isomorphism.

Theorem 1. *Let $p \equiv 3 \pmod 4$ be a prime with $p > 3$. Let ψ be a non-trivial character of $\mathrm{Cl}(F)$, where $F = \mathbb{Q}(\sqrt{-p})$. Then there exists a weight one harmonic modular form whose image under ξ_1 is $g_\psi(z)$ and whose holomorphic Fourier coefficients $r_\psi^+(n)$ vanish when $\chi_p(n) = 1$ or $n < -\frac{p+1}{24}$ and have the form*

$$r_\psi^+(n) = -\beta \sum_{\mathcal{A}\in\mathrm{Cl}(F)} \psi^2(\mathcal{A}) \log|u(n,\mathcal{A})|,$$

where $u(n,\mathcal{A})$ are units in $\mathcal{O}_H$ when $n \leq 0$ and algebraic numbers in H when $n > 0$. Here $\beta \in \mathbb{Q}$ depends only on p. Furthermore, for all $\mathcal{A},\mathcal{C} \in \mathrm{Cl}(F)$ we have that

$$\sigma_{\mathcal{C}}(u(n,\mathcal{A})) = u(n,\mathcal{A}\mathcal{C}^{-1})$$

and $\mathrm{N}_{H/\mathbb{Q}}(u(n,\mathcal{A}))$ is an integer with

$$-\tfrac{\beta}{2}\log|\mathrm{N}_{H/\mathbb{Q}}u(n,\mathcal{A})| = R_p^+(n) \tag{6}$$

when $n \neq 0$.

The statements about $u(n,\mathcal{A})$ bear a striking similarity to Stark's Conjectures on special values of derivatives of L-functions (see [8],[9]). When $n \le 0$ they are actually consequences of known cases of these conjectures. Also, (6) can be interpreted as a harmonic version of Dirichlet's identity (5).

The proof of this result makes use of the Rankin-Selberg method developed in [5] for computing heights of Heegner divisors. However, it requires the use of weight one harmonic forms with polar singularities in cusps in place of weight one Eisenstein series hence uses regularized inner products. In fact, modular curves of large prime level N and their Heegner divisors of height zero are used order to get to the individual Fourier coefficients.

As was pointed out by Zagier, his identity with Gross for the norms of differences of singular values of the modular j-function can be expressed in terms of the coefficients $R_p^+(n)$ given in (4). For simplicity, let $-d<0$ be a fundamental discriminant not equal to $-p$. and set $F' = \mathbb{Q}(\sqrt{-d})$. As is well-known, the modular j-function is well-defined on ideal classes of F and F' and takes values in the rings of integers of their respective Hilbert class fields. Also, values of the j-function at different ideal classes are Galois conjugates of each other. For any $\mathcal{A} \in \mathrm{Cl}(F)$ define the quantity

$$a_{d,\mathcal{A}} := \prod_{\mathcal{A}' \in \mathrm{Cl}(F')} (j(\mathcal{A}) - j(\mathcal{A}')), \tag{7}$$

whose norm to F is thus $\prod_{\mathcal{A}\in\mathrm{Cl}(F)} a_{d,\mathcal{A}}$ and is an ordinary integer. The result of Gross and Zagier [4, Theorem 1.3] is that this integer can be expressed in terms of $R_p^+(n)$ as follows:

$$\log \prod_{\mathcal{A}\in\mathrm{Cl}(F)} |a_{d,\mathcal{A}}|^{2/w_d} = -\tfrac{1}{4}\sum_{k\in\mathbb{Z}} \delta(k)\, R_p^+(\tfrac{pd-k^2}{4}), \tag{8}$$

where w_d is the number of roots of unity in F' and $\delta(k) = 2$ if $p|k$ and 1 otherwise.

There are two proofs of this factorization in [4]. One proof is analytic and the other one algebraic. The algebraic approach actually gives the factorization of the ideal $(a_{d,\mathcal{A}})$ in $\mathcal{O}_H$ for each class $\mathcal{A} \in \mathrm{Cl}(F)$. It is enough to state it for the principal class $\mathcal{A}_0$. Suppose that ℓ is a rational prime such that $\chi_p(\ell) \ne 1$. Then the ideal (ℓ) factors in $\mathcal{O}_H$ as

$$\ell = \prod_{\mathcal{A}\in\mathrm{Cl}(F)} \mathfrak{l}_{\mathcal{A}}^{\delta(\ell)}. \tag{9}$$

The $\mathfrak{l}_{\mathcal{A}}$'s are primes in H above ℓ uniquely labeled so that $\sigma_{\mathcal{C}}(\mathfrak{l}_{\mathcal{A}}) = \mathfrak{l}_{\mathcal{A}\mathcal{C}^{-1}}$ for all $\mathcal{C} \in \mathrm{Cl}(F)$ and complex conjugation sends $\mathfrak{l}_{\mathcal{A}}$ to $\mathfrak{l}_{\mathcal{A}^{-1}}$. It is shown in [4] that the order of $a_{d,\mathcal{A}_0}$ at the place associated to the prime $\mathfrak{l}_{\mathcal{A}}$ is given by

$$\mathrm{ord}_{\mathfrak{l}_{\mathcal{A}}}(a_{d,\mathcal{A}_0}) = \tfrac{1}{2}\sum_{k\in\mathbb{Z}} \delta(k) \sum_{m\ge 1} r_{\mathcal{A}^2}\left(\tfrac{pd-k^2}{4\ell^m}\right). \tag{10}$$

Our second main result gives a modular interpretation of the individual values $|a_{d,\mathcal{A}}|$. It is convenient to give it as a twisted version of (8).

Theorem 2. *For any harmonic form given in Theorem 1 having holomorphic Fourier coefficients $r_\psi^+(n)$ and $-d<0$ any fundamental discriminant different from $-p$ we have*

$$\sum_{\mathcal{A}\in \mathrm{Cl}(F)} \psi^2(\mathcal{A}) \log |a_{d,\mathcal{A}}|^{2/w_d} = -\tfrac{1}{4}\sum_{k\in\mathbb{Z}} \delta(k)\, r_\psi^+(\tfrac{pd-k^2}{4}) \tag{11}$$

where $a_{d,\mathcal{A}}$ is defined in (7).

Since $p>3$ is a prime number, $\mathrm{Cl}(F)$ has odd order $H(p)$ and ψ^2 is only trivial when ψ is trivial. Thus we can express each logarithmic term in the sum on the left hand side of (11) in terms of the holomorphic coefficients of a harmonic modular form associated to a theta series.

As with Theorem 1, this is proved using the methods of [5] and not the analytic technique of [4], which uses the restriction to the diagonal of an Eisenstein series for a Hilbert modular group. In particular, we make use of a real-analytic function $\Phi(z)$ that transforms with weight 3/2 and level 4, and use holomorphic projection to obtain an equation between a finite linear combination of $r_\psi^+(n)$'s and an infinite sum, similar to one in [5]. We also use machinery from [6]. A novel feature is an elementary counting argument needed to construct a Green's function evaluated at CM points. Actually, equation (11) is a particular example of a more general identity involving values of certain Borcherds lifts.

Numerical examples and the fact that it would also imply (10) motivate the following.

Conjecture. *In Theorem 1 we have:*

(i) The number $2/\beta$ is an integer dividing $24H(p)h_H$, where h_H is the class number of H.

(ii) For $\mathcal{B}\in \mathrm{Cl}(F)$, let $\mathfrak{l}_\mathcal{B}$ be a prime ideal above the rational prime ℓ as in (9). Then the order of $u(n,\mathcal{A})$ at the place of H corresponding to $\mathfrak{l}_\mathcal{B}$ is

$$\mathrm{ord}_{\mathfrak{l}_\mathcal{B}}(u(n,\mathcal{A})) = \tfrac{2}{\beta}\sum_{m\geq 1} r_{(\mathcal{A}^{-1}\mathcal{B})^2}\left(\frac{n}{\ell m}\right).$$

At all other places of H, $u(n,\mathcal{A})$ is a unit. In particular, $u(n,\mathcal{A})\in \mathcal{O}_H$ for all $n,\mathcal{A}$.

Finally, we remark that further numerical calculations suggest that results analogous to Theorem 1 and the Conjecture should hold for non-dihedral newforms. Again, see [3] for details.

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